

ON RIBBON \mathbb{R}^4 'S

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ABSTRACT. We consider ribbon \mathbb{R}^4 's, that is, smooth open 4-manifolds, homeomorphic to \mathbb{R}^4 and associated to h -cobordisms between closed 4-manifolds. We show that any generalized ribbon \mathbb{R}^4 associated to a sequence of h -cobordisms between non-diffeomorphic 4-manifolds is exotic. Notion of a positive ribbon \mathbb{R}^4 is defined and we show that a ribbon \mathbb{R}^4 is positive if and only if it is associated to a sequence of stably non-product h -cobordisms. In particular we show that any positive ribbon \mathbb{R}^4 is associate to a subsequence of the sequence of non-product h -cobordisms from [BG].

It is well known that there are examples of pairs of homeomorphic but not diffeomorphic simply connected closed smooth 4-manifolds, see [K]. It is also known that any such a pair of homeomorphic, smooth, simply connected and closed 4-manifolds is h -cobordant, [W]. Equivalently, given such a pair of non-diffeomorphic 4-manifolds, each one can be obtained from the other one by a regluing of a certain open smooth 4-manifold, usually called a “ribbon \mathbb{R}^4 ”, [DF] or [K]. A ribbon \mathbb{R}^4 used in a such reimbedding can be obtained from the h -cobordism and is homeomorphic but not diffeomorphic to the standard Euclidean four-space, \mathbb{R}^4 . So, it is an example of what is usually referred as an *exotic* \mathbb{R}^4 . Any h -cobordism between a pair of smooth, possibly diffeomorphic, 4-manifolds may be used to construct a ribbon \mathbb{R}^4 , but it is not known whether each of them is necessarily exotic. This paper provides a partial answer to the question which ribbon \mathbb{R}^4 's are exotic. We are working under an assumption that the given h -cobordism can not be turned into a product cobordism by blowing up both of its boundary components finitely many times (see Definition 5).

Let $(W^5; M_0^4, M_1^4)$ be an h -cobordism between two non-diffeomorphic, oriented, smooth, closed, simply connected 4-manifolds. After trading handles if necessary, we may assume that W has a handlebody description with only 2- and 3-handles:

$$W \cong (M_0 \times I) \cup \left(\bigcup_{i=1}^k h_i^2 \right) \cup \left(\bigcup_{j=1}^k h_j^3 \right),$$

where I is the unit interval, $I = [0, 1]$, and the matrix of incidence numbers between 2- and 3-handles is the identity matrix. These incidence numbers are equal to the intersection numbers in the middle level of the cobordism, $M_{1/2} =$

$\partial((M_0 \times I) \cup (\bigcup_{i=1}^k h_i^2) - M_0$, between the *belt* (or *dual*) spheres of the 2-handles and the *attaching* spheres of 3-handles, see [RS]. We denote the attaching spheres by A_i and the belt spheres by B_i , $1 \leq i \leq k$. (Often the belt spheres are called “descending spheres”, and attaching spheres are called “ascending spheres”.) Both the attaching spheres and the belt spheres are families of disjointly embedded 2-spheres in $M_{1/2}$, but beside k intersection points of $A_i \cap B_i$, $1 \leq i \leq k$, recorded in the intersection matrix, there may be some additional intersection points between the attaching and the belt spheres. These extra intersection points on any $A_i \cap B_j$ can be grouped into pairs with opposite signs. Note that in the absence of these extra pairs of intersections the 2- and 3-handles in W form complementary pairs of handles that can be removed from the handlebody decomposition. In that case there is a product structure for W , that is, a diffeomorphism $W \cong M_0 \times I$. In our situation the h -cobordism has no smooth product structure, so there has to be at least one extra pair of intersections between A_* and B_* .

We denote by X a regular neighborhood in the middle level of the union of these spheres, $X = \mathcal{N}(A_* \cup B_*)$. Extra pairs of intersections result in $\pi_1(X)$ being nontrivial. We can use Casson’s construction [C] to cap the generators of $\pi_1(X)$ by Casson handles inside $M_{1/2} - \text{int}X$. Casson’s construction may produce new pairs of intersections between A_* and B_* , but when considered separately, each family of spheres remains disjoint. The boundary components of W , M_0 and M_1 in our notation, can be obtained by surgering the middle level, $M_{1/2}$. These surgeries are performed on A_* spheres to obtain M_1 and on B_* spheres to obtain M_0 . Surgering X produces two compacts in the boundary components, Y_0 and Y_1 . It follows from Freedman’s work [F] that M_0 and M_1 are homeomorphic and that the cobordism W is homeomorphic to the product cobordism, $M_0 \times I$. Since we have assumed that M_0 and M_1 are not diffeomorphic, W can not be diffeomorphic to the product cobordism. Following [DF], [FQ] or [K] we may assume that W is smoothly product over the complement of a the compact Y_0 in M_0 . Note that $\partial Y_0 = \partial Y_1 = \partial X$ and so M_1 can be obtained from M_0 by replacing Y_0 by Y_1 , that is $M_1 \cong (M_0 - \text{int}Y_0) \cup_{\partial Y_0 = \partial Y_1} Y_1$. Also, $M_{1/2} \cong (M_0 - \text{int}Y_0) \cup_{\partial} X$.

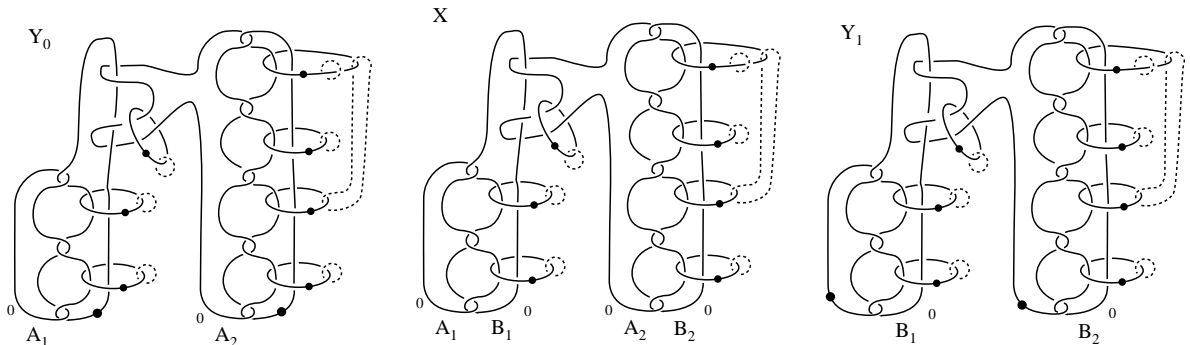


FIGURE 1

A link calculus description of an example of such Y_0 , X and Y_1 is presented in Figure 1. Dashed circles are generators of the fundamental groups. A compact obtained like Y_0 or Y_1 in an h -cobordism is known to be diffeomorphic to a complement in the 4-ball of an embedded disc that spans a ribbon link in the boundary of the 4-ball. We will refer to such a compact Y_0 as a *ribbon complement associated to the h -cobordism W* or simply as a *ribbon complement*, when the cobordism is determined by the context. By inverting the cobordism W we obtain an h -cobordism $(\overline{W}; M_1, M_0)$ and Y_1 is a ribbon complement associated to \overline{W} . Meridians to the components of the bounding ribbon knot or link generate the first homology group of a ribbon complement. If we use these meridians (with 0-framings) to attach the standard 2-handles to a ribbon complement, the resulting manifold is the standard 4-ball. If the standard 2-handles are replaced with Casson handles and the remaining boundary is removed, then the resulting open 4-manifold is homeomorphic to the Euclidean four-space, \mathbb{R}^4 , and is called a *ribbon \mathbb{R}^4* [DF]. In the case that a ribbon \mathbb{R}^4 is not diffeomorphic to \mathbb{R}^4 , we refer to it as an *exotic ribbon \mathbb{R}^4* . If Casson handles are attached to ribbon complement associated to W ambiently in $M_0 - \text{int}Y_0$, then we say that the resulting ribbon \mathbb{R}^4 is *associated to the cobordism W* . An example of an exotic ribbon \mathbb{R}^4 associated to a non-product h -cobordism was explicitly described in [B]. Although only two Casson handle were involved in its construction, the number of their kinks grow so fast with the level that the description, as I. Steward [S] has politely phrase it, “verges on bizzare”. To obtain a simpler exotic ribbon \mathbb{R}^4 , a sequence of non-product h -cobordisms was used in [BG] in the following way. Let $R_* = \text{int}Y_* \cup_{i=1}^m CH_i$ be a ribbon \mathbb{R}^4 built from a ribbon complement Y_* . For every positive integer n , we denote by U_*^n the open manifold built by attaching to Y only the first n levels of each of the Casson handles CH_i , $U_*^n = \text{int}Y_* \cup_{i=1}^m (CH_i)^n$. Suppose that each U_*^n is associated to an h -cobordism $(W_n; M_{n,0}, M_{n,1})$ in the sense that Y_* is associated to W_n and the n level open Casson towers $(CH_i)^n$ are embedded ambiently into $M_{n,0} - \text{int}Y_*$. Then we say that R_* is *associated to the sequence of cobordism $\{W_n\}$* . Note that if R_* is a ribbon \mathbb{R}^4 associated to an h -cobordism W then R_* is also associate to the sequence of cobordisms $\{W_n = W\}$.

To continue we introduce a model of a ribbon complement. We start as before by first constructing a compact X that is a regular neighborhood of A_* and B_* spheres in the middle level of an arbitrary h -cobordism W . It is always possible, although not in a unique way, to group the extra pairs of the intersections so that each can be obtained by a finger move of an A_* through a B_* . In other words, we may introduce finger moves on the regular neighborhood $\mathcal{N}(\coprod_k (S^2 \vee S^2)) \subset \sharp_k(S^2 \times S^2)$ so that the result is diffeomorphic to X . This construction produces a distinguished set of generators for $\pi_1(X)$ consisting of loops embedded into ∂X . If we retreat the fingers emanating from A_* so that their tips are only tangent to B_* , we call the remaining generators *accessory loops*. When the fingers are returned to their initial

positions one of the two intersection points of each finger is designated for accessory loops to pass through and we adjust accessory loops accordingly. To complete our set of generators for $\pi_1(X)$ we choose a loop for each finger that consists of an arc on A_* and an arc on B_* , both ending on the two intersection points on the finger. We call these loops the *Whitney loops*. Using isotopies when necessary, we assume that loops generating $\pi_1(X)$ are disjoint outside the extra intersections between A_* and B_* . After projecting X along the cobordism into M_0 or M_1 , only the interior of X has been replaced and the accessory and the Whitney loops are in M_0 or M_1 , respectively. It is easy to describe this projection in the terms of link calculus: to surger X into Y_0 0-framings of the link components representing the family B_* are replaced by dots, namely, 2-handles are replaced by 1-handles or by scooped out 2-handles. Note that $\pi_1(X)$ is a subgroup of $\pi_1(Y_0)$, the latter also includes generators that are meridians to the dotted circles that used to represent B_* in X . We will continue to call “accessory” and “Whitney” loops the generators for $\pi_1(Y_0)$ induced by the surgery.

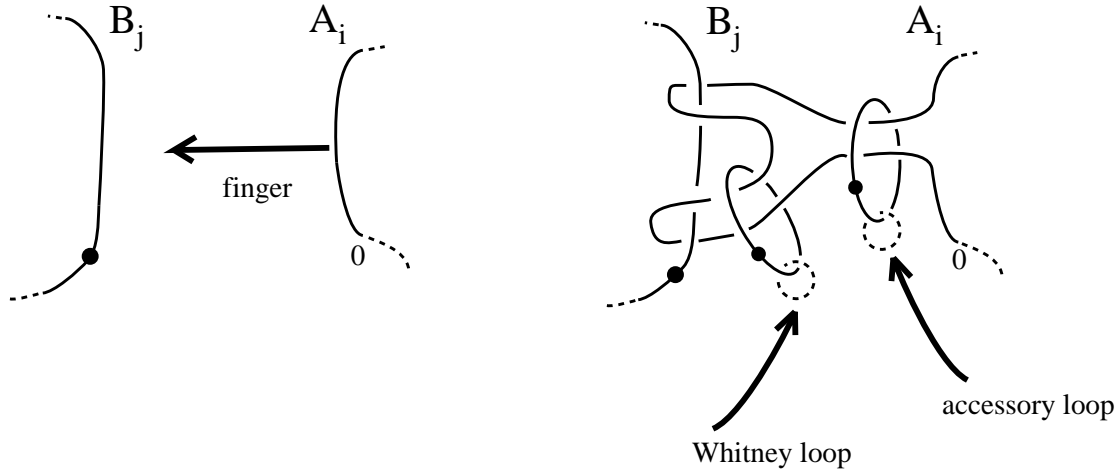
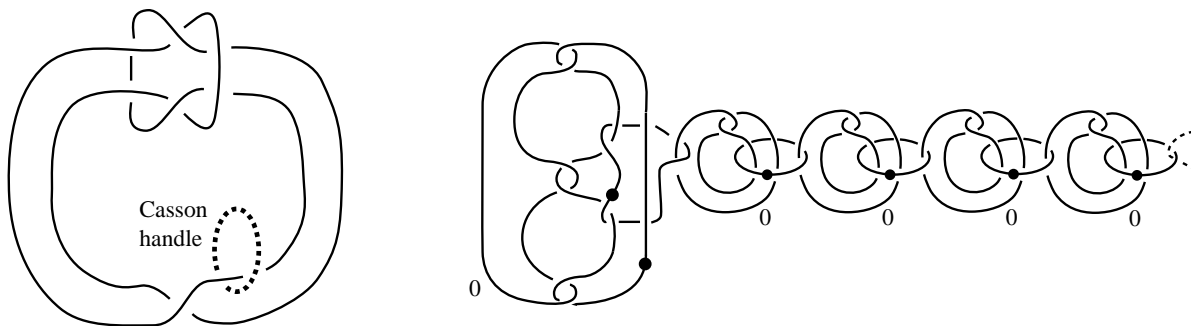


FIGURE 2

Figure 2 is a link calculus picture of a finger move. Note that the components of B_* are already surgered into dotted circles, the picture of X can be obtained by replacing dots on these components by 0-framings. We may build our model of a ribbon complement by adding such finger moves to a collection of Hopf links with a 0-framed and a dotted component in each (link calculus picture of complementary pairs of 1- and 2-handles), but often we will end up with more “accessory” dotted circles than needed. The extra dotted circles may be slid off their parallels and off our picture where they are removed. Alternatively, the “accessory” dotted circle from Figure 2 is added only when the finger closes a loop and introduces a new accessory generator of $\pi_1(X)$.

Remark. The construction of ribbon complements described above (and in [K] and [DF]) involves a specific family of ribbon links. The above mentioned exotic \mathbb{P}^4 from

FIGURE 3. R_0 – an example of a generalized ribbon \mathbb{R}^4 .

[B] is an example of an ribbon \mathbb{R}^4 , but the simplest known exotic \mathbb{R}^4 (introduced in [BG]) is not a “ribbon \mathbb{R}^4 ” although it has a ribbon knot (Figure 3) associated to it: to the meridian of the ribbon complement of the disc bounding ribbon knot from the left part of Figure 3 a single Casson handle is attached. The attached Casson handle has a single kink at each level and all its kinks are positive. This exotic \mathbb{R}^4 , which we denote by R_0 , is built from the same ribbon complement as the example from [B], but instead capping both the accessory and the Whitney loops by Casson handles, there are only one Casson handle and a standard 2-handle involved. We will consider a slightly more general situation and so we say that a contractible open smooth 4-manifold built from a ribbon complement by capping the accessory and Whitney loops with any combination of 2- and Casson handles is a *generalized ribbon \mathbb{R}^4* . The notion of such a generalized ribbon \mathbb{R}^4 being associated to a sequence of h -cobordisms can be defined exactly as before.

Two questions arise naturally. The first one is whether a ribbon \mathbb{R}^4 associated to an h -cobordism between non-diffeomorphic 4-manifolds (or to a sequence of such cobordisms) is necessary exotic. Conversely: which combinations of ribbon complements and attached Casson handles produce exotic ribbon \mathbb{R}^4 's. Answer to the first question was known to be positive in the case of a ribbon \mathbb{R}^4 associated to a single h -cobordism between non-diffeomorphic 4-manifolds, [K, pages 98 – 101]. This is also true in a slightly more general situation.

Theorem 4. (Compare with Theorem 3 in [K, page 98].) *A (generalized) ribbon \mathbb{R}^4 associated to a sequence of h -cobordisms between non-diffeomorphic 4-manifolds is exotic.*

Proof. Let $R = Y \cup (\partial Y \times (0, \infty)) \cup_{i=1}^m CH_i \cup_{j=1}^n H_j^2$ be a generalized ribbon \mathbb{R}^4 associated to a sequence of h -cobordism, $\{(W_n; M_{n,0}, M_{n,1})\}$, where $M_{n,0}$ and $M_{n,1}$ are not diffeomorphic and where H_j^2 denotes an open 2-handle, that is $(H_j^2, \partial H_j^2) \cong (D^2 \times \mathbb{R}, S^1 \times \mathbb{R})$. Assume that R is diffeomorphic to the standard \mathbb{R}^4 . Then, since the ribbon complement Y is a compact subset of R , there is a smooth 4-ball B_0 embedded in R that contains Y in its interior. The ball B_0 being compact is contained in U^k for some $k \geq 1$. W is a product over $M_{n,0} \cup Y$ and we use this

product structure to lift ∂B_0 into a smooth 3-sphere S_1 in $M_{k,1}$. An argument from [K] shows that S_1 has to bound a standard 4-ball in $M_{k,1}$: briefly, by embedding k level Casson towers of U^k into the standard 2-handles we construct a smooth embedding of U^k into the standard 4-ball which we consider to be in the standard 4-sphere, S^4 . The piece of the cobordism W_k over U^k can be transplanted into the product cobordism $S^4 \times I$. Recall that the complement of a smooth 4-ball in 4-sphere is also a 4-ball. We lift the complement of the 4-ball B_0 in $S^4 \times \{0\}$ to the top of the cobordism, $S^4 \times \{1\}$. Since this cobordism is a product over the complement of Y , the complement of the lifted 4-ball is a standard 4-ball, bounded by S_1 , and the cobordisms over the complements of $\text{int} B_0$ in S^4 and $M_{k,0}$ are product. So, the product structure over $M_{k,0} - B_0$ can be extended over B_0 , contradicting our assumption that $M_{k,0}$ and $M_{k,1}$ were not diffeomorphic. \square

One might expect that the second question has an equally simple answer, that all possible generalized ribbon \mathbb{R}^4 's are exotic. This is not the case: the simplest ribbon complement is diffeomorphic to $S^1 \times D^3$ and the ribbon link involved is the unknot. If any Casson handle is attached over the meridian and the boundary is removed, the resulting manifold is the standard \mathbb{R}^4 [F, page 381]. However, it is easy to describe this manifold as a generalized ribbon \mathbb{R}^4 , for example, replace one of the two Casson handles of R_1 in Figure 5 by a standard open 2-handle.

Definition 1. If L is an accessory loop of a ribbon complement Y_0 than the set of Whitney loops that intersect L is called the *Whitney set of L* .

Definition 2. If a signed tree associated to a Casson handle has a positive branch, than the Casson handle is *positive*. If there are more positive than negative edges emanating from every vertex of the associated tree, then the Casson handle is *strictly positive*.

Definition 3. Let Y be a ribbon complement and R a ribbon \mathbb{R}^4 obtained from Y by adding Casson handles. Suppose that there is an accessory loop such that every loop of its Whitney set is capped by a positive Casson handle and, in the case that its Whitney set contains only one loop, then the accessory loop itself is capped by a positive Casson handle. In the case that there are more than one loop in the Whitney set we require that this accessory loop coincides with at most one finger emanating from any A_* spheres. Then we say that R is a *positive ribbon \mathbb{R}^4* .

It is not known whether ribbon \mathbb{R}^4 's that are not positive are exotic or not. However, ribbon \mathbb{R}^4 's that are not positive can not be associated to a sequence of non-product h -cobordisms that satisfy the following additional assumption.

Definition 5. Let $(W^5; M_0^4, M_1^4)$ be an h -cobordism between two oriented, smooth, closed, simply connected 4-manifolds. We say that W is *stably non-product* if $M_n \# (\overline{CP}^2)$ and $M_{n+1} \# (\overline{CP}^2)$ are not diffeomorphic for any nonnegative integer n .

Remark. It is not known to the author whether there exists a pair of simply connected closed smooth 4-manifolds M_0 and M_1 that are homeomorphic and non-diffeomorphic and such that $M_0 \sharp n(\overline{\mathbb{C}P}^2)$ and $M_1 \sharp n(\overline{\mathbb{C}P}^2)$ are diffeomorphic for some $n \geq 1$.

Theorem 6. *A ribbon \mathbb{R}^4 associated to a sequence of stably non-product h -cobordisms is positive.*

We will prove that a ribbon \mathbb{R}^4 that is not positive can not be associated to a sequence of stably non-product h -cobordisms W_n by showing that at least one W_n can be turned into a product cobordism by blowing up its end sufficiently many times. A process of removal of double points is described in [Ku] and a short outline of that method is given next.

Suppose that $(\Delta, \partial\Delta)$ is an immersed disc in a 4-manifold $(N, \partial N)$ with a single double point in the interior of Δ . Furthermore, suppose that this double point is negative. After blowing up N we can replace Δ by an embedded disc in $(N \sharp \overline{\mathbb{C}P}^2, \partial(N \sharp \overline{\mathbb{C}P}^2))$ that spans the same loop in $\partial N \sharp \overline{\mathbb{C}P}^2 = \partial N$: Let E and E' represent two “exceptional curves”, i.e., copies of $\mathbb{C}P^1$ in general position and embedded in the added $\overline{\mathbb{C}P}^2$. If E and E' are equipped with opposite orientations then they intersect in a single point that has the positive sign. Choose a small ball centered at the intersection between E and E' , the intersection between the boundary of the small ball and E and E' will form a Hopf link. Similarly choose a small ball in the interior of N that is centered at the double point of Δ . The intersection between Δ and the boundary of the ball is again a Hopf link. The centers of these two balls can be connected by a path that avoids Δ , E and E' . Remove the intersection between the interiors of the balls and Δ , E and E' . Now the two Hopf links in the boundaries of the balls are connected by two pipes that follow the chosen path. The resulting disc represents the same second homology class as Δ and has the same boundary, but it is embedded into $(N \sharp \overline{\mathbb{C}P}^2, \partial(N \sharp \overline{\mathbb{C}P}^2))$.

Notice that this procedure can prune all the branches of a tree associated to a Casson handle that have a negative kink. Also, for every non-positive Casson handle there is a natural number k so that every brunch of the tree associated to the Casson handle has a negative kink on the first k levels. (Recall that a tree associated to a Casson handle has finitely many edges coming from every vertex.) Consequently, if we perform sufficiently many blow-ups of an ambient 4-manifold, we may replace any Casson handle that is not positive with an embedded standard 2-handle.

Proof of Theorem 6. Suppose that R is a non-positive ribbon \mathbb{R}^4 associated to a sequence of h -cobordisms W_n . Choose k large enough such that every branch of non-positive Casson handles contains a negative kink on the first k levels. We will work in W_n where we have embedded first k levels of the Casson handles from

R. After blowing up W_k sufficiently many times we may replace all non-positive Casson towers in $\tilde{W}_k := W_k \cup (\sharp \overline{\mathbb{C}P^2}'s \times I)$ with standard 2-handles so we may assume that all the remaining Casson towers have only positive kinks. We can use the embedded 2-handles to perform Whitney tricks and cancel pairs of 2- and 3-handles from the handlebody decomposition of \tilde{W}_k whenever possible. If none of the Casson handles of R is positive, this procedure removes all the 2- and 3-handle pairs and \tilde{W}_k has a product structure. In particular, W_k is not stably non-product.

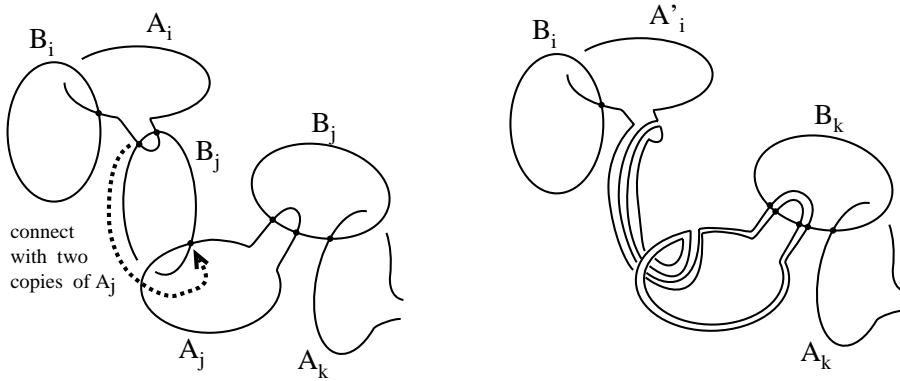


FIGURE 4

If there are no accessory loops we can use Norman trick from Figure 4 to remove the extra pairs of intersections. Working in the middle level we start from such an extra pair, say between A_i and B_j . Note that B_j and A_j have a single intersection point since otherwise there would be an accessory loop on them. Each of the two intersections is removed by Norman trick (see [FQ]) by removing a disc from A_i centered at the intersection point and a disc from a copy of A_j that is centered at an intersection point between A_j and B_j . Then the boundaries of these two removed discs are meridians to B_j and are connected by a tube, Figure 4. The resulting new A_i sphere, denoted by A'_i , intersect each B_* that A_j did, so there are four intersections between A'_i and B_k in our example from Figure 4. Repeating this process produces a cascade of fingers, each piercing a B_t such that A_t has no fingers. The application of Norman trick will add two copies of A_t to each finger that ends on B_t therefore removing the pair of intersections we have started with. Now all Whitney discs are removed and again, \tilde{W}_k has a product structure.

If Y does contain accessory loops, since we have assumed that R is not positive, each accessory loop ventures over more than one finger emanating from a single A_* or the Whitney set for the accessory loop contains a loop capped by a non-positive Casson handles. In the later case, after the blow-ups this Whitney loop is capped by the standard two handle and the finger containing it is removed, breaking the given accessory loop. So now we may assume that the only accessory loops remaining are those that contain more than one finger emanating from a single A_* . Starting with two such fingers and pushing them over other spheres from A_* we produce cascades

of fingers. The accessory loop is closed when both cascades of fingers intersect the same B_* . We consider such a loop, emanating from, say, A_i and ending on B_j . Fingers that may start from A_j also can be grouped in pairs, each ending on a same B_* . Following these pairs we can not close a loop by having a pair of fingers ending on B_i , otherwise we could select a member from each pair and obtain an accessory loop that passes over at most one finger emanating from any A_* . In that case the non-positiveness would apply that at least one of the associated Whitney loops can be capped by the standard 2-handle. Consequently we may assume that our pairs of fingers emanating from A_i ends on a B_j such that A_j has no fingers and no extra intersection points. Using the Norman trick as before the pairs of intersections on fingers are removed. So all the extra pairs of intersections can be removed and again, \tilde{W}_k has a product structure. \square

The converse to Theorem 6 is also true, every positive ribbon \mathbb{R}^4 is associated to a sequence of stably non-product h -cobordisms.

Theorem 7. *Every positive ribbon \mathbb{R}^4 can be associated to a subsequence of the sequence $\{W_m\}_{m=2}^\infty$ of stably non-product h -cobordisms constructed in [BG].*

Each of these h -cobordisms from [BG] we denote here by $(W_m; M_{m,0}, M_{m,1})$, where $M_{m,1} \cong E(m) \sharp k(\overline{\mathbb{C}P}^2)$ and $M_{m,0}$ decomposes as a connected sum of $\mathbb{C}P^2$'s and $\overline{\mathbb{C}P}^2$'s. For simplicity we have not included “ k ” or \tilde{W}_m in our notation and $E(m)$ denotes the minimal elliptic surface with no multiple fibers and of the Euler characteristic $12m$.

Proof. First we will show that for each positive ribbon \mathbb{R}^4 , $R = \text{int}Y \cup_i CH_i$ and for each natural number k we can embed $U^k = \text{int}Y \cup_i (CH_i)^k$ in some $M_{m,1}$, when m is large enough and $M_{m,1}$ contains sufficiently many copies of $\overline{\mathbb{C}P}^2$. Each of this embeddings factors through an embedding into a compact obtained from the closure of U_0^k (that is, the first k levels of R_0 from Figure 3) by adding extra 2-handles and parallel copies of the Casson tower, shown in Figure 7. Then we show that the h -cobordism obtained by regluing the embedded ribbon complement Y is diffeomorphic to W_m .

According to Definition 3 each positive ribbon \mathbb{R}^4 contains an accessory loop whose Whitney set is capped by positive Casson handles. We start our construction by embedding all the other Casson handles into the standard 2-handle and each positive Casson handle capping a loop from the fixed Whitney set is embedded into the CH^+ , the positive Casson handle with one kink per level. Now each positive ribbon \mathbb{R}^4 is embedded into a ribbon \mathbb{R}^4 that has a single accessory loop, similar to one of those in Figure 5. In this figure there is a sequence of ribbon \mathbb{R}^4 's, R_n , $n \geq 1$, each but the first one is built by attaching n copies of the Casson handle CH^+ to a ribbon complement which we denote by Y_n . Note that in R_1 the accessory loop is also capped by CH^+ , and for $n \geq 2$, the accessory loop is capped by the standard

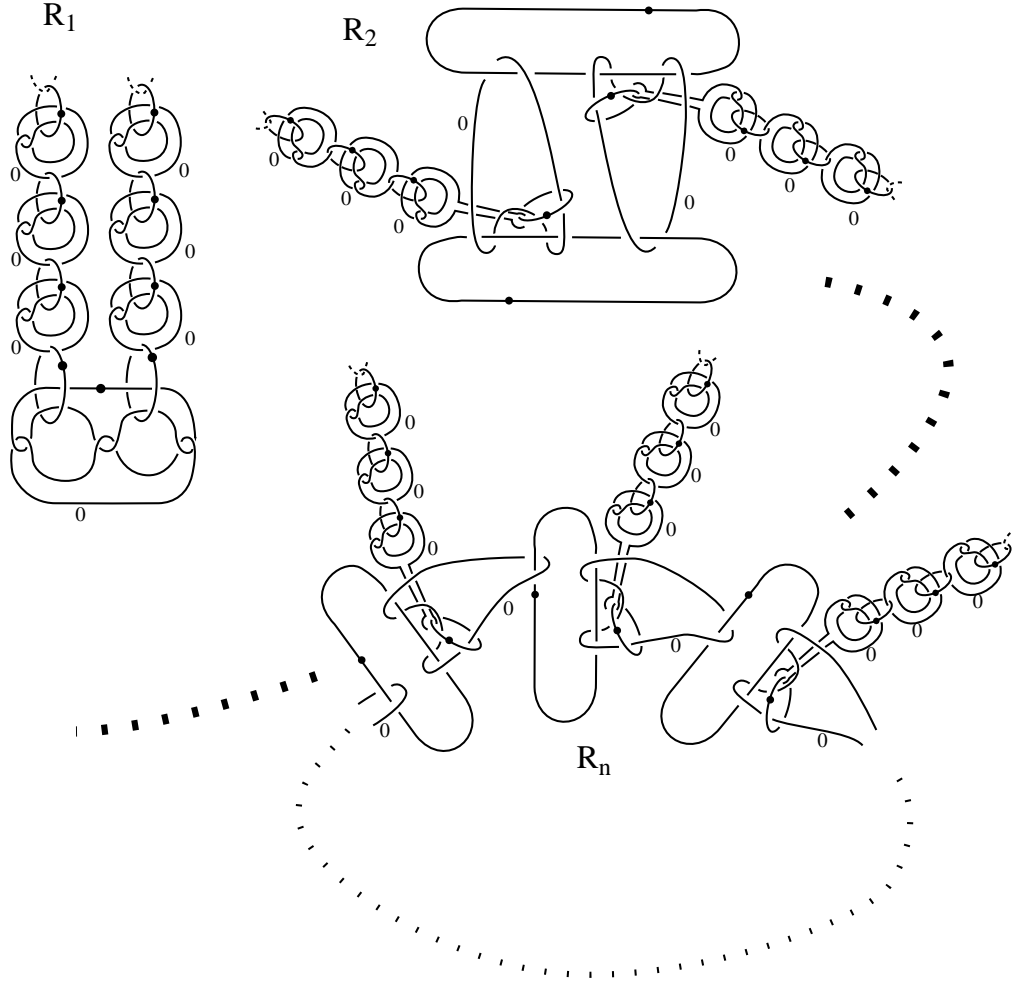


FIGURE 5

2-handle and its dotted circle disappears (compare with Figure 2). We denote by R'_n the resulting ribbon \mathbb{R}^4 in which we have embedded R . The index n in “ R'_n ” or “ R_n ” is equal to the number of pairs of A and B spheres that form the underlying middle level compact, X_n .

The middle level compact X_n , and therefore the ribbon complement Y_n , are uniquely defined up to isotopy by listing the geometric and algebraic numbers of intersections between A_* and B_* spheres. Furthermore we can isotope one link calculus picture of Y_n into another by sliding the 2-handles and dotted circles whose meridians are Whitney loops over the dotted circle corresponding to the accessory loop. Equivalently, the possible differences between link calculus pictures may occur as different choices of clasps, positions of dotted circles whose meridians are Whitney loops and twists of parallel strands in 2-handles. In Figure 6 it was shown how to deal separately with each of these differences. Since in any stage we are allowed to blow-up the ambient manifold finitely many times we can always introduce positive twists by attaching a -1 -framed 2-handle and sliding other 2-handles off it. Figure 6. So we may assume that all clasps between handles corresponding to A

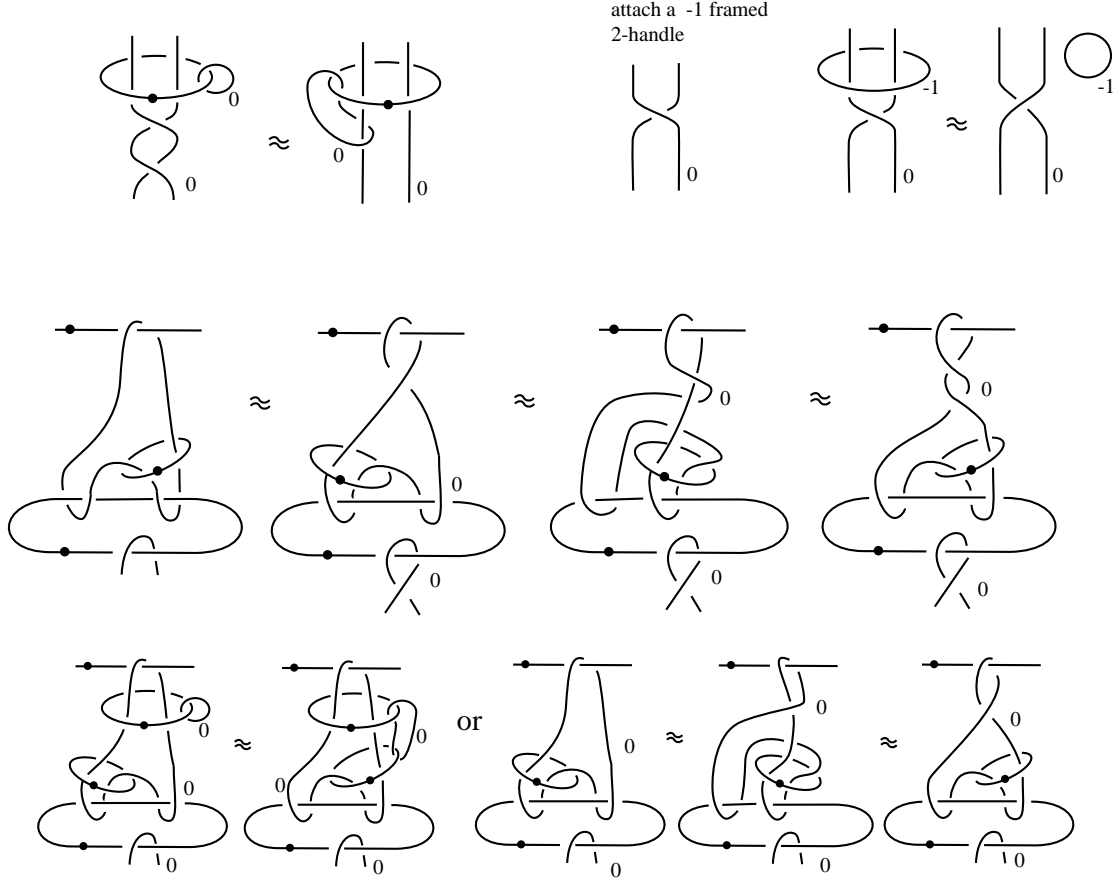


FIGURE 6

and B_* spheres and positions of the dotted circles corresponding to Whitney loops in a link calculus picture of R'_n are exactly the same as in the picture of R_n in Figure 5. The only possible difference remaining is in accessory loops. Therefore, our construction produces an embedding of a positive ribbon \mathbb{R}^4 into a possibly blown-up ribbon \mathbb{R}^4 that we have denoted by R'_n and that is built by attaching copies of the Casson handle CH^+ onto Whitney circles of Y_n , and by dealing with the accessory loop in the same fashion as in R_n . If we fix as generators for $\pi_1(Y_n)$ the Whitney circles w_1, \dots, w_n and the accessory loop a from Figure 5, then in general a' , the accessory loop of R'_n , is a word in these generators involving w_i 's. As before U_n^k will denote $Y_n \cup_{w_*} n(CH^+)^k \cup_a h^2$, where $(CH^+)^k$ is the Casson tower equal to the closure of the first k levels of the Casson handle CH^+ . Copies of $(CH^+)^k$ are attached over the Whitney loops w_* and the 2-handle h^2 is attached over the accessory loop a . Similarly, we define $(U')_n^k$ by attaching the 2-handle over a' instead of a .

We claim that we can extend the embeddings from [BG] of U_0^k into $M_{m,1}$, where m is large enough, to an embedding of the handlebody C_1 from Figure 7. The embeddings we are extending were described in [B], Figures 19 – 60, and in [BG], Figures 40 – 81. The modifications we have to add is to have an arbitrary number of Casson k -level towers (instead of only one in U_0^k) and arbitrary numbers of -1

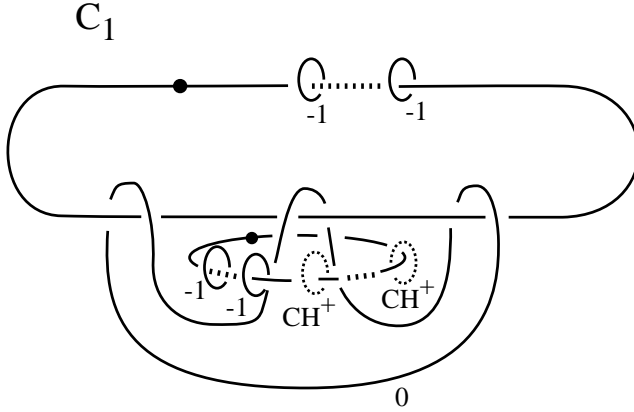


FIGURE 7

framed 2-handles linked with the two dotted circles in C_1 , Figure 7. To embed -1 -framed 2-handles linked with the larger dotted circle we follow Figures 40 – 43 and 49 – 54 in [BG]. In Figure 40 from [BG] the meridian of the larger dotted circle from our Figure 7 corresponds to the circle denoted by β . The meridian of the smaller dotted circle in Figure 7 we can follow in Figures 39 – 44 in [B], but the difference in our case that the actual pictures are the mirror images of those in [B] and the largest 0-framed two handle and the dotted circle have to switch their roles. So we read Figure 44 in [B] that each meridian of the smaller dotted circle from our Figure 7 is isotoped to a pair of meridians of the larger dotted circle from Figure 7. Each pair has one unlinked 0-framed component and the other one is -1 -framed and linked with all the second components of the other pairs. After passing these meridians into the other part of the manifold we have to take their mirror images so the framings in Figure 44 from [B] are now correct as drawn. Figure 60 from [B] shows how to deal with (now $+1$ -framed) linked second components of the pairs. Note that in figures from [BG] these pairs of meridians also are isotopic to β . In all cases we are left to cap 0-framed isotopes of the circle β which we can do by either Casson towers of arbitrary levels (Figure 59 in [B] or Figure 81 in [BG]) or we can slide them over the linked -1 -framed 2-handles and produce embeddings of -1 -framed 2-handles in our Figure 7. Each of these processes uses -1 -framed 2-handles isotop to β (Figure 81 from [BG]) end to procure them in a sufficient quantity we need to choose m large enough.

Next we construct an embedding of U_n^k , $n \geq 1$, into the handlebody C_1 from Figure 7. An embedding of U_1^k was described in [BG], Figure 47, namely a 0-framed 2-handle is added to connect the accessory and Whitney dotted circle, the result is U_0^k , but with two parallel Casson k level towers so it embeds in C_1 . To embed U_n^k , $n \geq 2$, into C_1 we connect the dotted circles corresponding to B spheres by $n - 1$ 2-handles. Figure 8 depicts this process in the case of U_2^k and U_3^k .

Figure 9 shows how to embed each U_n^k , $n \geq 1$, in C_1 . We start with C_1 and add complementary pairs of 2- and 3-handles (which corresponds in link calculus

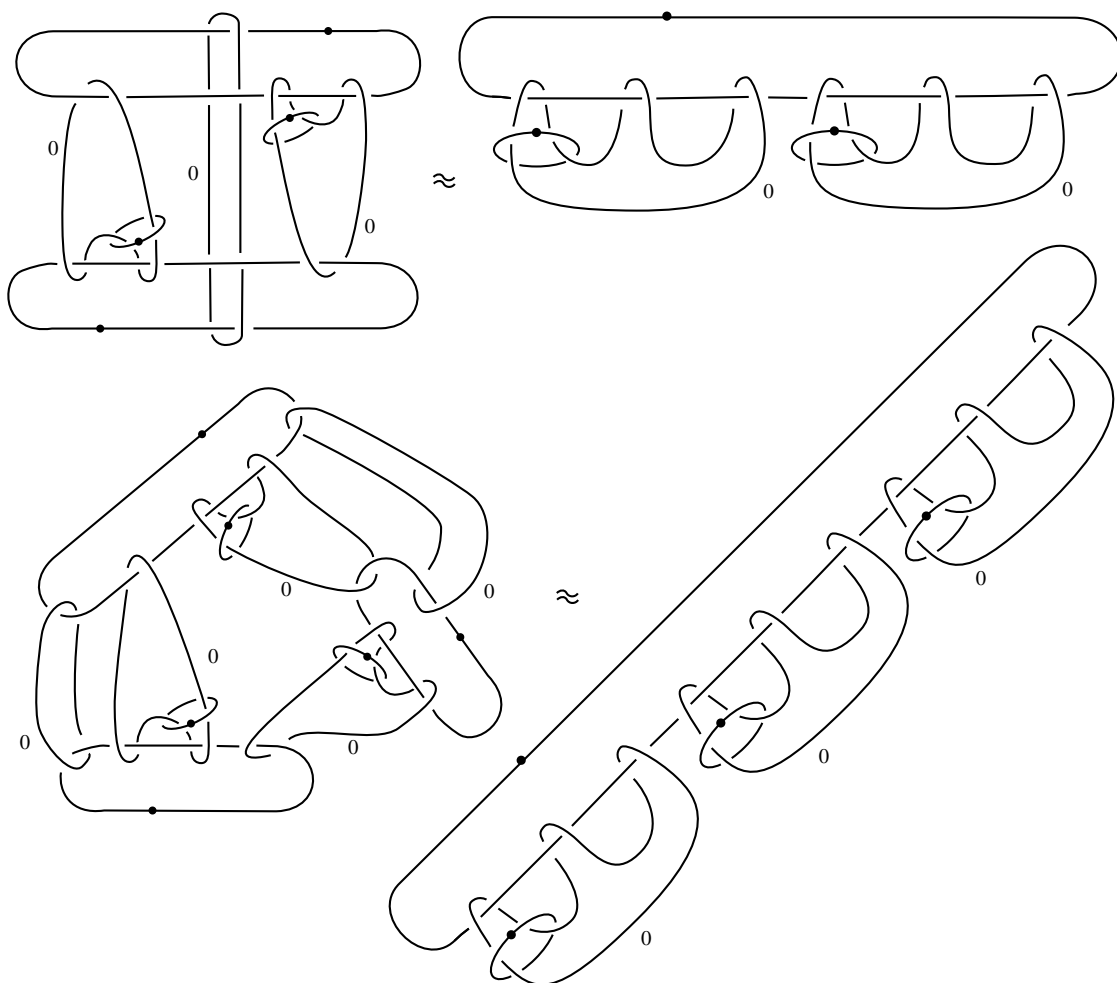


FIGURE 8

to adding unlinked unknots with framings 0) and then we slide the 2-handles over the 2-handle of C_1 . Next we perform isotopies to separate these parallel copies of this 2-handle. Figure 9 shows how to complete an embedding of U_2^k into C_1 and all the other U_n^k 's are embedded in the same fashion.

Next we show how to modify these embeddings to embed $(U'_n)^k$ into C_1 . Recall that the difference between $(U'_n)^k$ and U_n^k is only in the 2-handle capping the accessory loop. In the case of $(U'_n)^k$ the attaching circle of this 2-handle can also link other handles, but we can isotop this circle to be a word in our fixed Whitney circles, w_1, w_2, \dots, w_n . Figure 10 shows how to unlink the accessory loop from the dotted circles corresponding to Whitney loops. For each piece of the accessory loop that links once a Whitney dotted circle we add a -1 -framed 2-handle, as shown in Figure 10. Then we can use the added handle to slide the accessory loop off the dotted circle. The result of such a slide increases the framing of the accessory loop by $+1$. After we slid the accessory loop off all dotted circles the resulting new accessory loop is linked with only one dotted circle, as in the case of U_n^k , but its framing will be in general positive. We can slide in Figure 10 the dotted circle and

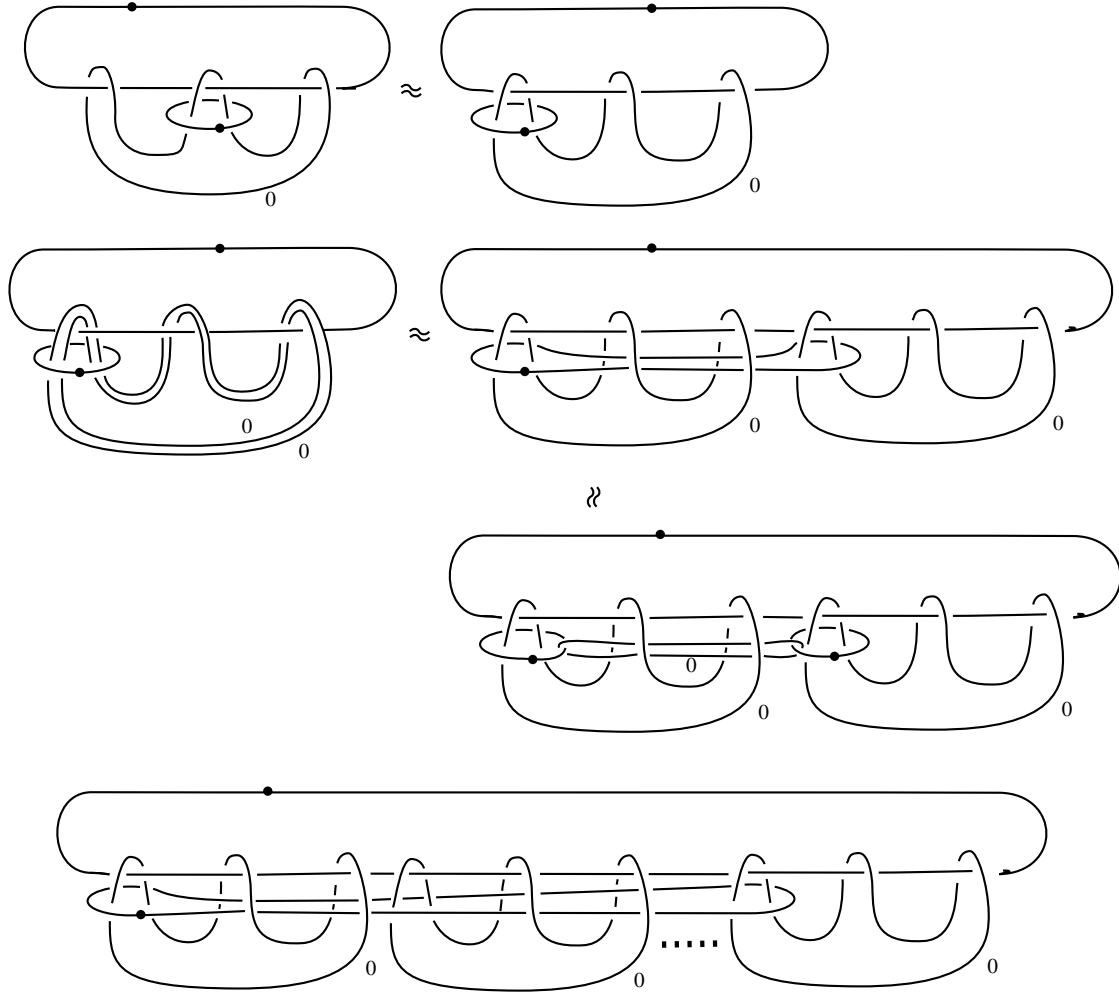


FIGURE 9

the N -framed accessory circle linked to it such that the dotted circle ends up linked with the visible 0-framed 2-handle. Then we can slid each of the two strands of the 0-framed handle over the N -framed handle and finally, we slid this (still 0-framed) 2-handle $2N$ times over dotted circle to unlink it from the N -framed handle. The result of this handle slides will render the canceling pair with the dotted circle and the N -framed 2-handle unlinked from anything else and therefore it can be removed from the picture. The other induced change is that the two strands of the 0-framed 2-handle have obtained N positive twists. The framing of the accessory loop can always be increased by any positive amount: attach a -1 -framed 2-handle as in Figure 10 and slide off it the 0-framed handle. This process blows up once the ambient manifold and introduces an extra positive twist. By using such a blow-up if necessary and reversing the handle slide, we assume that N , the framing of the accessory circle, is an even positive integer.

Our present link calculus pictures of $(U'_n)^k$ and R'_n differ from U_n^k and R_n in that accessory loop is capped by an unknotted 2-handle that has framing N instead of 0, where N is an even positive integer or, equivalently, one of the 0-framed 2-

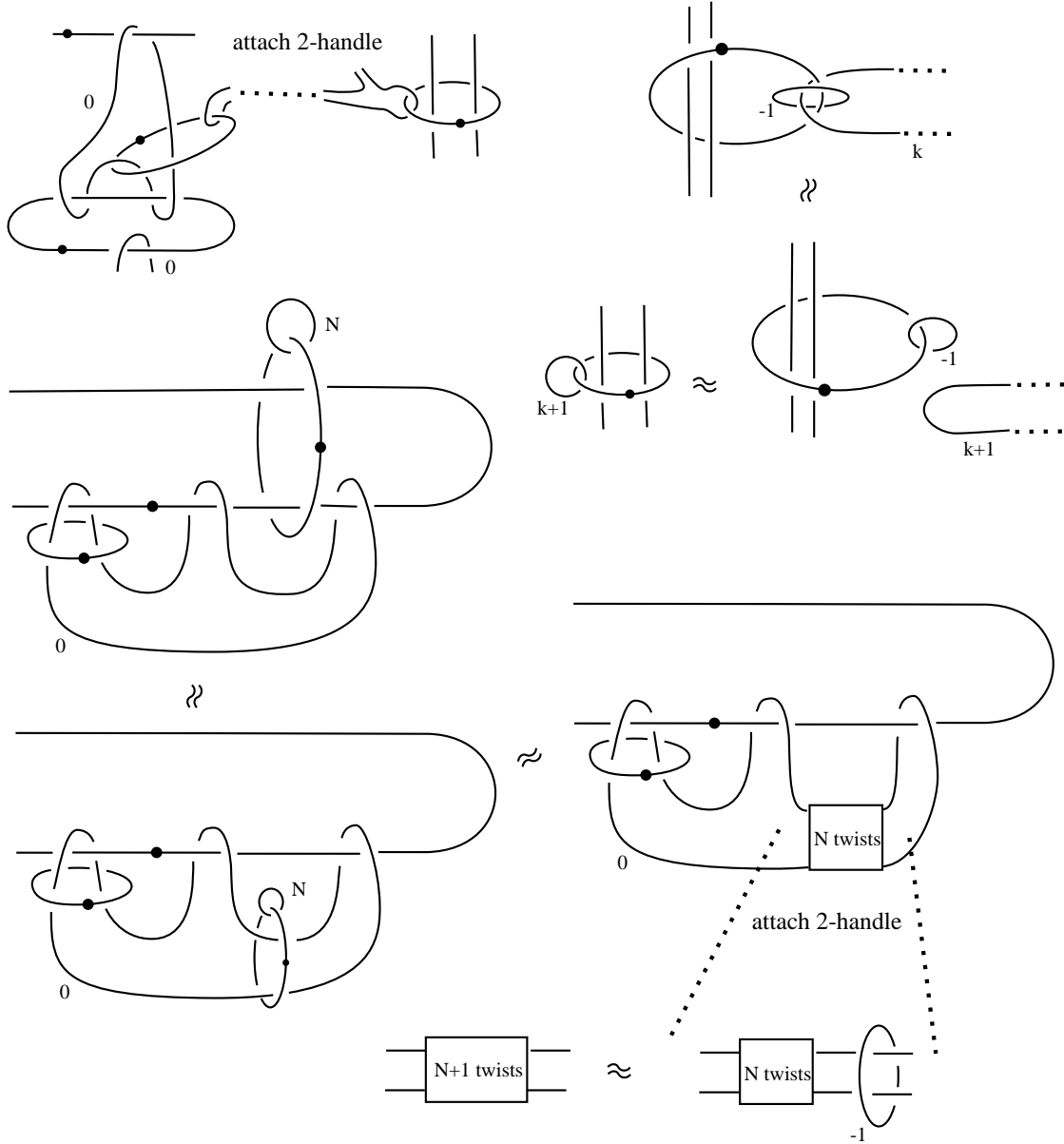


FIGURE 10

handles has N positive twists, see Figure 10. Figure 11 shows how to embed $(U')_n^k$ into C_1 by accommodating each pair of twists. Adding a pair of complementary 1- and 2-handles such that the 2-handle is 0-framed and linked twice with the dotted circle is equivalent of having two positive twists and so we add $N/2$ such pairs. The attaching of -1 -framed 2-handles as in Figure 11 replaces each of $N/2$ pairs of twists by a -1 -framed 2-handle that is meridian to the larger dotted circle in the picture of C_1 and so we have an embedding of $(U')_n^k$ into $M_{m,1}$.

Our next task is to obtain an h -cobordism from each embedding of $(U')_n^k$ into $M_{m,1}$. The other boundary component, $M_{m,0}$, is obtained by a reembedding of $(Y')_n \subset (U')_n^k \subset M_{m,1}$ that switches the roles of 0-framed 2-handles and appropriate dotted circles in Figures 8 and 9. In particular, Figure 9 is replaced by Figure

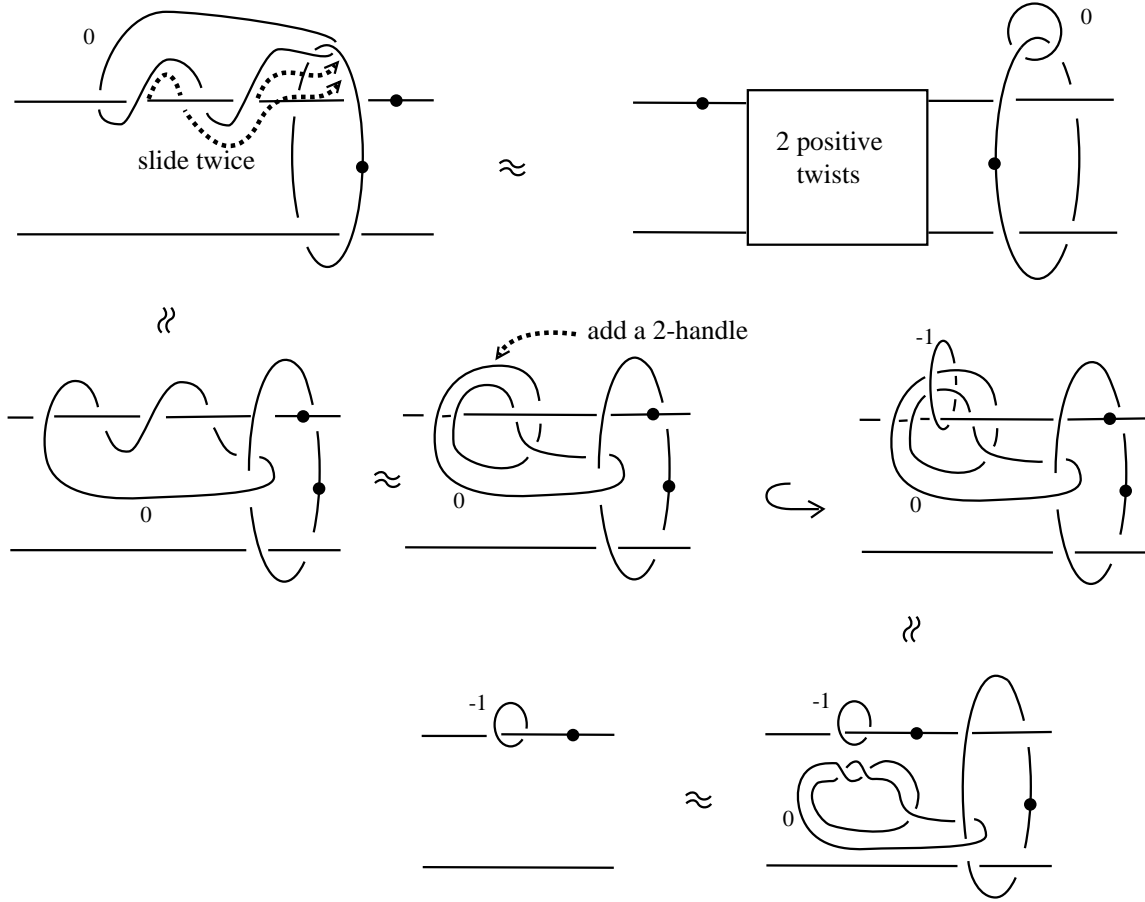


FIGURE 11

12. The 0-framed Hopf links added in Figure 12 were complementary pairs of 1- and 2-handles in Figure 9.

We claim that the obtained h -cobordisms are $(W_m; M_{m,0}, M_{m,1})$, that is, they are in the same sequence of h -cobordisms obtained by the reimbeddings of $Y_0 = Y_1 \subset U_1^k \subset M_{m,1}$, [BG], but with possibly different m and the number of blow-ups corresponding to a given number of levels, k . The boundary components of the cobordisms in [BG] were constructed by regluing of the Mazur rational ball that is visible in our figures as a subhandlebody of C_1 , Figure 7. The two embeddings of the Mazur ball were using its dual handlebody decomposition and we will do the same here with embeddings of C_1 . We will explicitly show the embedding of Y_2 into $M_{m,0}$ and from the construction it will be clear how to obtain the embeddings of $Y_n, n \geq 3$.

The top part of Figure 13 recapitulates the embedding of Y_2 into C_1 . There is a 3-handle attached over unlinked 2-handle, normally not visible in a link calculus picture. Below is a link calculus picture of the dual handlebody decomposition. We will follow a convention from [BG] and now we present its shout outline. To obtain a link calculus picture of a dual decomposition from a given link calculus picture one may start by drawing the mirror image of the given link calculus picture. Then

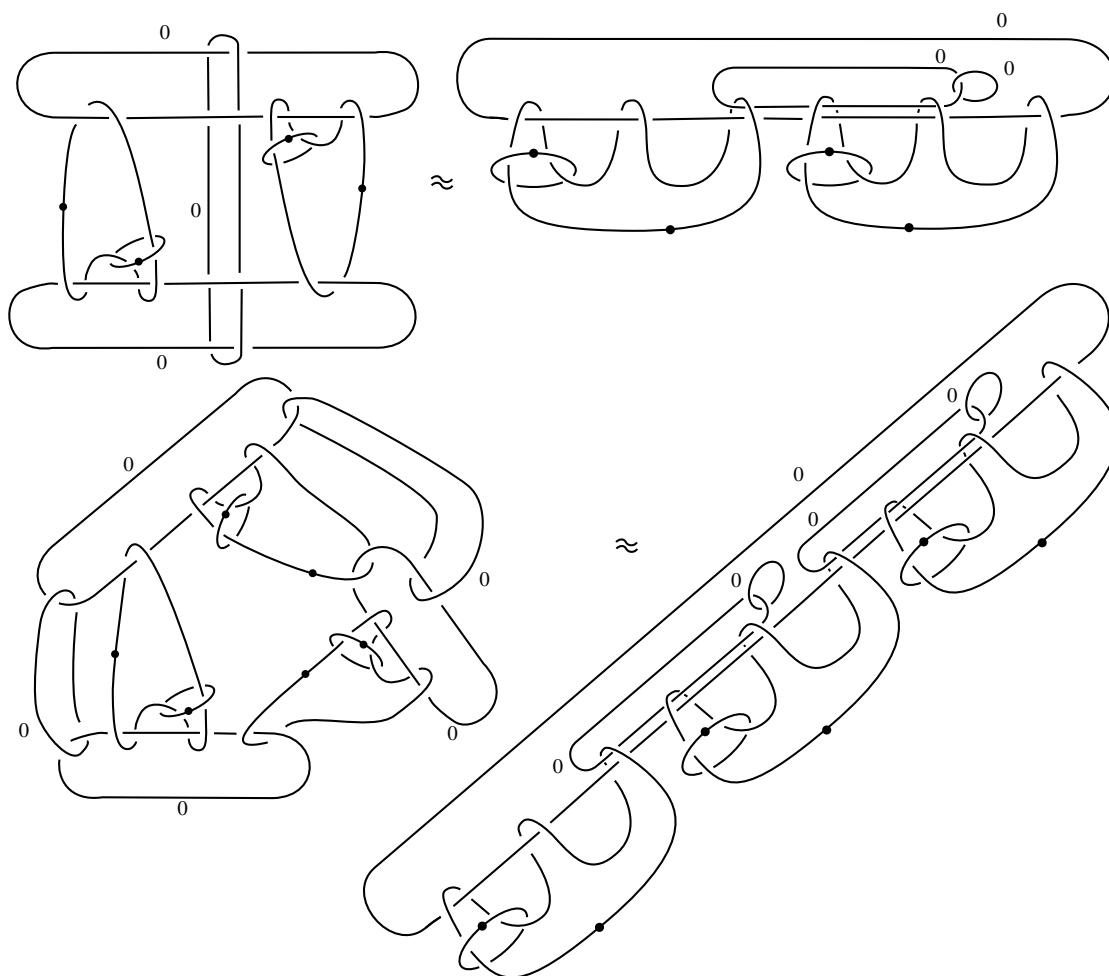


FIGURE 12

the dotted circles (that is, 1-handles or, equivalently, scooped out 2-handles) obtain (0)-framings and the signs of all others framings are changed and enclosed by parentheses, see [BG]. Next, to each link component that was an attaching circle of a 2-handle in the original link calculus picture one attaches a 0-framed 2-handles over the meridian of the component, see [K]. (Recall that in a dual decomposition of a 4-dimensional handlebody 1-handles become 3-handles and vice versa and the 0-handle becomes a 4-handle.) Such a link calculus picture contains components marked with a dot (1-handles), components with an integer framing (2-handles) and components whose framings are integers enclosed by parentheses. A handlebody described by a such link has two boundary components: the “ ∂^- -component” is obtained by performing (a 3-dimensional) surgery only on components in parentheses, and the other boundary component, the “ ∂^+ -component”, is the result of the surgery of all the components of the link.

To facilitate the description we have labeled all components of the dual part of Figure 13 by capital letters, A – I. Now we describe a diffeomorphism between the two pictures of the dual decomposition. The handlebody to the left also contains

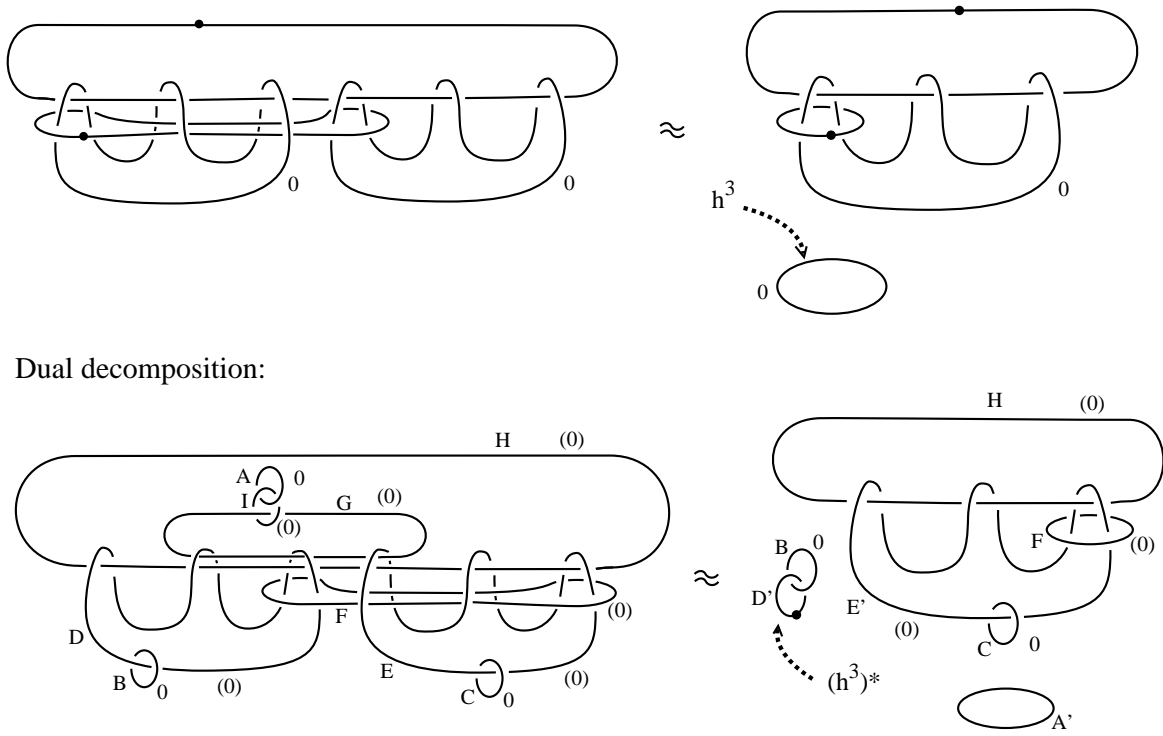


FIGURE 13

a 4-handle and three 3-handles, the duals of the 1-handles in the original decomposition, namely they have to be attached over the components D, E and G. Also there is a 1-handle that has to be added to the ∂^- -boundary component that is not visible in this picture. First we slide D and E over I and off G. The new D' and E' are now linked with A, and A can be slid off I over G. Components I and G are unlinked and can be removed from the picture. Note that now A occupies the place previously occupied by G. A is then slid of D and E over B and C, respectively. The resulting 2-handle, now denoted by A' is unlinked from the rest of the components, but the 3-handle that used to be attached over G is now attached over A' and together they form a complementary pair of 2-and 3-handles. Next, we slide D' over E' and the result is visible in the lower right corner of Figure 13. Now the 1-handle is visible; it coincides with D' and together with the 2-handle B it forms a complementary pair. Now we have two complementary pairs of handles that we remove from the picture. By adding where appropriate Casson towers and -1 -framed 2-handles we obtain C_1 .

We proceed with a description of an embedding of Y_2 into $M_{m,0}$. The top of Figure 14 reproduces from Figure 12 the link calculus picture of Y_2 with two 0-framed 2-handles added. Below is a link calculus picture of its dual decomposition. As before we have labeled all the components of this link. Again we have an invisible 1-handle, a 4-handle and three 3-handles attached over D, E and F. First we slide B over E and then twice over C to unlink it from H. The resulting component, B', is unlinked from the rest of the components, but it coincides with the 2-handle

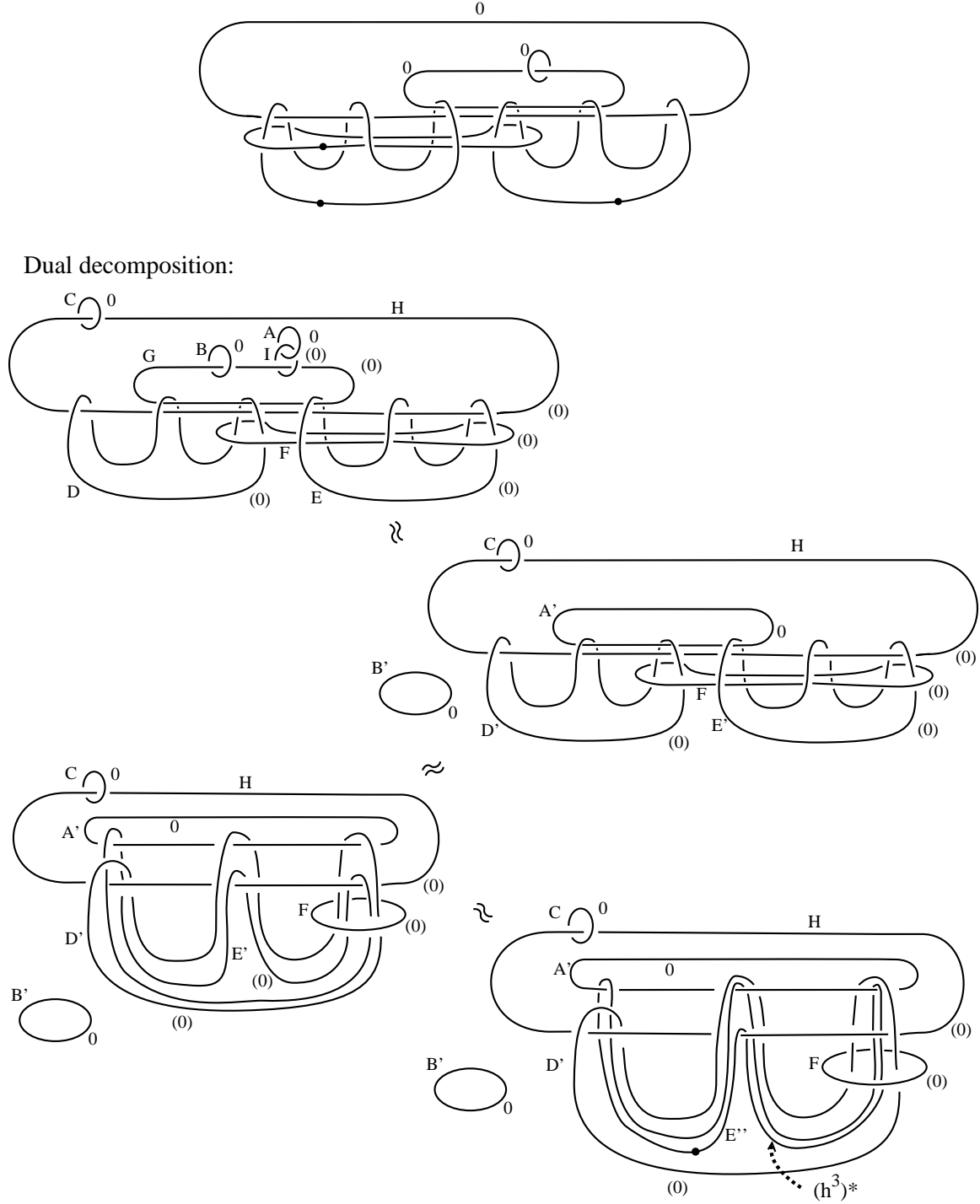


FIGURE 14

attached over E . Similarly as above, we remove the components G and I and the resulting link calculus picture is visible in Figure 14 and the next picture in that figure is the result of an ambient isotopy of the link. Then we slide E' over D' and the resulting component, E'' , is where we add (from “below”, to the ∂^- -component of the boundary) the missing 1-handle. The 3-handle that was incident only with E is now incident with both E'' and B' and 3-handle originally attached to D is now incident with D' and E'' .

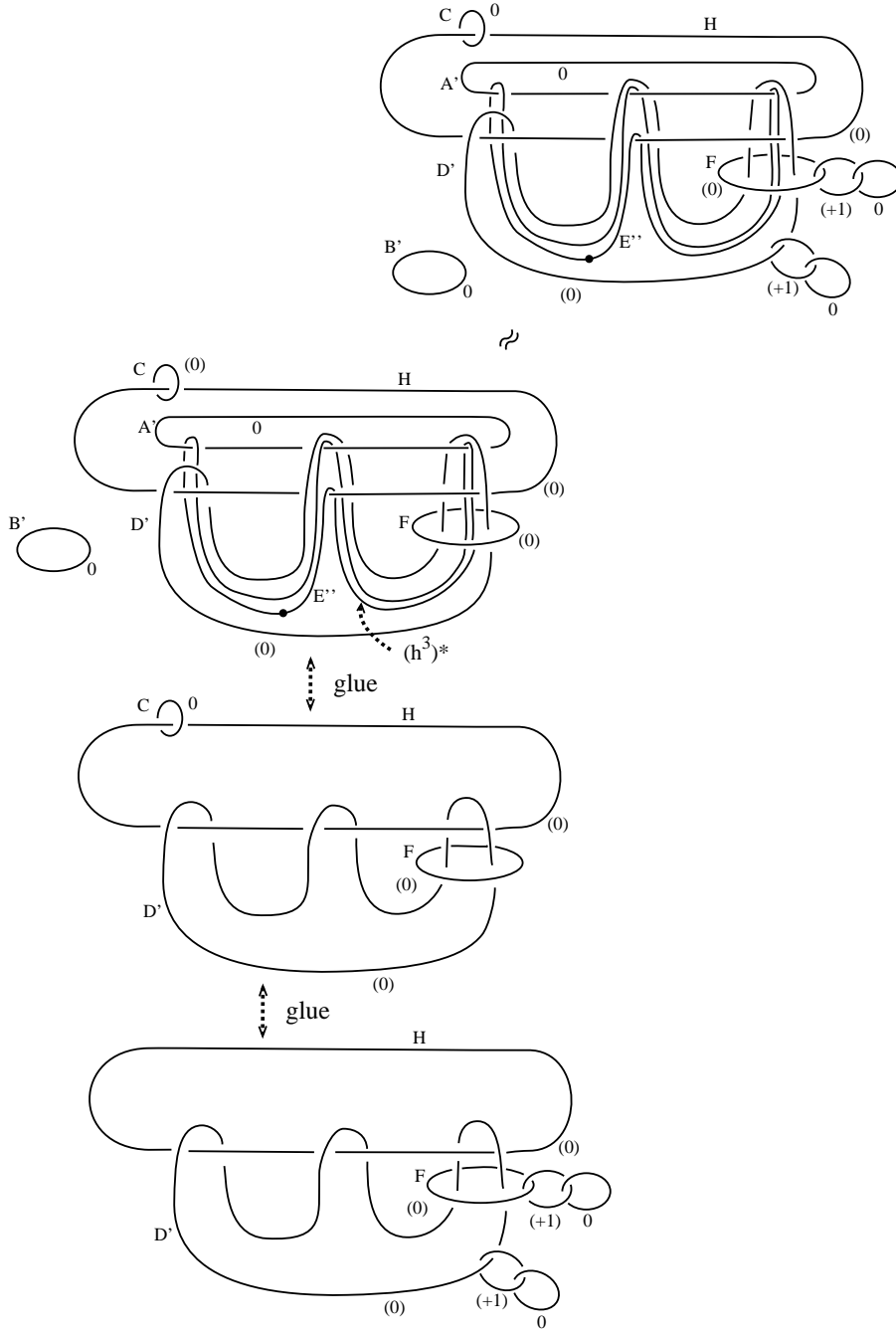


FIGURE 15

The ∂^- -component of the boundary of our handlebody is the same as in the case of embedding into $M_{m,1}$ and, after adding two -1 -framed 2-handle we have a handlebody from Figures 44 and 45 in [BG]. The result of this addition is in Figure 15. Furthermore we decompose this manifold as a union of three pieces stack over each other and glued over appropriate boundary components. The bottom piece in Figure 15 is the dual picture of adding two -1 -framed 2-handles. Its ∂^+ boundary component is obtained by surgeries on D' , F and H . Then, we add C , a 0 -framed 2-handle. Now, the ∂^+ boundary component of this middle piece is S^3 and the difference between the handlebody we are considering and the one used in

embedding of Y_2 into $M_{m,1}$ from [BG] is in the pair of 1-handle and a 0-framed 2-handle added in the piece on the top and glued by a diffeomorphism of the standard 3-sphere. Since by changing a gluing diffeomorphism of the standard 3-sphere we can not change the smooth structure of the resulting 4-dimensional manifold, we have only to consider the diffeomorphism type of the piece on the top. It is easy to see that this piece, together with 3-handles and the 4-handle is diffeomorphic to the standard 4-ball. Namely, by sliding D' over (0)-framed C we can unlink all the (0)-framed components and then E'' and A' form a complementary pair of 1- and 2-handles. Now the 3-handles are attached over the resulting (0)-framed components, and on top of them we have to attach the 4-handle. Therefore, the handlebody from Figure 15, together with invisible 3- and 4-handles is the same as the manifold from Figures 44 and 45 in [BG] and the argument there shows how to decompose the obtained boundary component of the h -cobordism, $M_{m,0}$, into a connected sum of $\mathbb{C}P^2$'s and $\overline{\mathbb{C}P}^2$'s. To generalize to embeddings of Y_n , $n \geq 3$ into $M_{m,0}$ note that the difference will be in having extra canceling pairs of 1- and 2-handles that are again separated from the most of other handles of $M_{m,0}$ and, as in the case of Y_2 , can be removed from the picture.

To complete the proof, we have to deal with the changes necessary to embed more general $(U')_n^k$ into $M_{m,0}$. Again we can use the trick from Figures 10 and 11, by switching the roles of the biggest 0-framed and dotted circles in the lower half of Figure 10 and by replacing the dot by 0-framing of the two horizontal line segments throughout Figure 11. Since all handle slides were of 2-handles, the only difference is that we are sliding 2-handles over 1- and 2-handles rather than 2-handles over only 1-handles and so we do not have to use the forbidden link calculus moves involving slides of dotted circles over components without a dot. \square

Note that although we have seen that any positive ribbon \mathbb{R}^4 can be associated to a subsequence of the same sequence of stably non-product cobordisms W_m from [BG], the particular m needed to embed a given U^k into $M_{m,1}$ and $M_{m,0}$, and the number of their blow-ups, both depend on the given ribbon \mathbb{R}^4 .

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